Rolling Dice for Fun and Profit

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November 29, 2008

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1 Revision History

The canonical version of this paper is `http://www.nimlabs.org/~nim/nov/report.pdf`

Revision 1.2	Nov 29 2008.	Add Last Rolls.
Revision 1.1	Nov 29 2007.	Add multi-turn strategies.
Revision 1.0	Nov 11 2007.	First release version.
Revision 0.1	Nov 29 2006.	Super rough draft for last night.

2 Disclaimer – THESE NUMBERS ARE USELESS

The numbers and conclusions in this paper are *not* based on the full rules of the Games. Several rules are omitted for simplicity and computability. You should not attempt to use these numbers to make conclusions about the actual Games.

Seriously. Even if the results presented were entirely correct (which they are not), playing "optimally" is no fun at all, few people do it, and many people will mock you if you try.

No, really. I'm not shitting you. These numbers are useless. They are interesting and fun to think about but that is as far as it goes. I started this project to learn Common Lisp, not to produce useful results.

3 Introduction

Since the dawn of time, man has been playing the Games. In this paper, I study several aspects of Game strategy. I have written a library — the October Framework — for Games strategy computation and have investigated several simple strategies for specific points in play.

The most significant piece of the framework is `OptStrategy` which, given a roll, returns the action (dice to reroll) with the highest expected value as defined by an arbitrary value function. The value function is defined over turn stopping states, and the optimization is thus per turn.

`OptStrategy` also takes a hash function used to memoize results. Every state that hashes to the same value must have the same expected value and result. A more complete description of the framework is in §7.

4 Single Turn Strategies

Finding the optimum strategy for a single turn value function is very easy in The October Framework. Several basic value functions result in useful building block strategies.

4.1 Optimal Points Per Turn

Over the years, several researchers have examined how to maximize expected points per turn. This question is very easy to answer with the October framework by using an `OptStrategy` with a value function that simply returns the stopping score. The hash function for this strategy need only encode the held score – it doesn't matter how you got to where you are in the turn, it only matters how many points you have held and how many dice you are rolling. The results from this approach with the framework are in agreement with the conclusions from other researchers.

The expected value per turn of this strategy is about 340.9045. The probability distribution of keeping a particular score (P_{keep}) when playing this strategy is shown in Figure 1. The full decision table of the strategy is shown Appendix A. An abbreviated, and more useful for playing, version is shown in Table 1.

From the probability distribution of scores on a single turn (see Figure 1), we can compute the probability distribution for multiple turns. Figure 2 shows the expected score for 5, 10, and 15 successive turns under the assumption that a player stops rolling after keeping a score of 5000 or more.

One can also compute the expected score for a large number of rolls, ignoring the end of game condition. Figure 3 shows the distribution of points per turn for 100 turns. By the central limit theorem, this should approach a normal distribution. However even after 1000 turns the distribution is notably non-normal. Since in any given game a player will likely not have that many turns in which to roll, approximation with a normal distribution seems unwise.

4.2 Not Being Come On – Rolling At Least N Points

One of the questions that comes up most in Games is the question of what score to go out at. Although there is no actual penalty for being come on, for social reasons most players choose to wait for a going out score with very low chances of a Come On.

To address this question I created an `OptStrategy` with a value function which returns 1 if stopping nets more than a threshold number of points, 0 otherwise: `AtleastStrategy`. When the threshold is set to a given score, the expected value of this strategy is the probability of a single turn resulting in at least the threshold score.

Roll	**Held Points**	**Roll**	**Held Points**
5 dice		*4 dice*	
1xxxx	0–800	1xxx	50–100
11xxx	0	15xx	50
115xx	0	5xxx	50–300
15xxx	0–500	55xx	50–100
155xx	0	*3 dice*	
5xxxx	0–850	5xx	100
55xxx	0–550		

Always stop rolling unless your situation is listed above.
Always reroll the most possible dice, keeping the highest scoring die.
Note that there are no entries for two dice: you always stop or convert after rolling two dice.

Table 1: Description of Optimal Points Per Turn Strategy

When the threshold score is set to the amount needed for a Come On, the expected value is the probability of a Come On in the worst case scenario for the going out player: a competitor trying their hardest for a Come From Behind. Observation shows that this worst case is frequently realized. Figures 4 and 6 show this probability for a range of going out margins.

From these data it is trivial to compute the upper bound of the probability of being Come On. Given N competitors each with score $O[i]$ and a going out score of S, the worst case probability of being Come On is:

$$P_{\text{co}} = 1 - \prod_{i}^{N} \left(1 - P_{\text{atleast}} \left(S - O[i]\right)\right)$$

Appendix D has P_{atleast} values in tabular form.

Observation shows that most skilled players have a good intuitive grasp of these numbers. Most players seem to target $< 1\%$ odds of being come on when "assassinating" a game and some tolerate probabilities above 5% when ending a high scoring long running game.

The full decision table of the Atleast strategy is shown Appendix in B. The strategy to have the highest odds of reaching a large score (a threshold of more that 1600) is fairly simple: always reroll the fewest dice possible, unless your roll is in `(15xx, 55xx, 11xxx, 15xxx, 55xxx, 115xx, 155xx)` in which case reroll the most dice possible.

4.3 Rolling Exact Scores

Occasionally a player will be targeting an exact score. For example, to achieve the Perfect score. To investigate this, I created an `OptStrategy` with a value function that is 1 only for the exact target score and 0 otherwise. This strategy optimizes the chance of hitting the target exactly. The expected value of this strategy with the same value function is the probability that it succeeds and hits the target. This probability is shown in Figures 5 and 6, and Appendix E has the same data in tabular form.

5 Multi-Turn Strategies

In order to optimize a strategy across multiple turns, I added another layer of recursion: the value function for a turn-end state recursively calls itself on the subsequent turn-start state. By defining the base case to be the situation when the strategy keeps a score that ends the game, `OptStrategy` is able to optimize across the whole game.

Unlike single turn strategies, multi-turn strategies must have the current position on the `Scoreboard` included in the hash table of memoized results. Additionally, the maximum recursion depth must be considerably higher to encompass many more rolls than in a single turn (see §7.1). Both of these drastically increase the size of the stored hash data and the computation time required.

Although the multi-turn strategies and their expected values are computed deterministically by walking the entire tree of game play, calculating exactly the distribution of number of turns taken would require inordinate CPU time. Instead, I sampled the distribution using the `PlayGames` functionality; actually simulating games at the die-roll level using the PRNG and counting how many turns elapsed. After one million samples, the distributions for each of the strategies appear to have converged.

5.1 Unconstrained

Perhaps the simplest multi-turn strategy is to minimize the number of turns until the end of the game without constraint. Both going out and rolling to Perfect are separate end cases to examine. In some sense, these could be considered the optimal opening and mid-game strategies, but they do not at all address end-game strategy.

The optimal turns until going out strategy is very similar to the optimal points per turn strategy, only differing on a few rolls and only near the going out score. The sampled distribution of number of turns until

	Moments				Gamma	
Strategy	mean	variance	skew	kurtosis	a	b
To Going Out						
Unconstrained	15.63	19.40	0.264	0.0891	11.8	1.33
No-Wimp	15.78	20.25	0.274	0.103	11.5	1.37
Foo	22.98	119.2	1.05	1.72	4.56	5.04
Oof	19.91	32.26	0.144	-0.123	11.16	1.78
Aggressive $\lambda = .9$	15.67	20.64	0.319	0.152	11.2	1.40
Aggressive $\lambda = .75$	15.89	23.37	0.389	0.228	10.2	1.55
Aggressive $\lambda = .5$	18.78	55.88	0.762	0.883	6.24	3.01
Aggressive $\lambda = .25$	263.68	4.77e4	1.72	4.42	1.51	174
To Perfect						
Unconstrained	18.42	26.02	0.412	0.391	12.6	1.46
No-Wimp	18.58	26.77	0.416	0.381	12.5	1.49
Low Foo	31.25	238.5	1.23	2.57	4.36	7.17
High Foo	60.73	1304	1.44	3.43	3.05	19.9

Table 2: Moments and parameters for multi-turn strategy distributions of number of turns taken.

end-game for both going out and Perfect appear somewhat normal although, like the single turn strategy, not quite normal enough to allow useful approximation.

Figure 7 shows the distribution of turns along with the normal distribution. Table 2 shows the curve parameters compared with some constrained strategies.

5.2 Foo

One popular strategy — Foo — is to only keep scores that result in a warp. Since this requires knowledge of where on the board one is, it must be considered as a multi-turn strategy.

5.2.1 Foo Until You're Out

There are several variations on Foo, relating to the end condition of the optimization. The simplest case is to play until your score is over the going out score without regard to other players or end-game dynamics.

To optimize this case, I used a value function that minimized the expected number of turns until the going out score was achieved. The value function returns a large negative value ("*badness 10000*") whenever keeping a score would result in non-Foo play, effectively forcing the strategy to play Foo.

The distribution of number of turns taken, produced by sampling, bears a striking resemblance to the Gamma distribution. It is not clear whether this is coincidental or if the Gamma parameters obtained by curve fitting have a meaning in terms of game probability structure.

Figure 8 shows the distribution along with the best-fit gamma distribution. Table 2 shows the fit parameters, alongside other multi-turn strategies.

5.2.2 High and Low Foo to Perfect

Since end-game dynamics are very separate from the strategies being considered, more useful strategies for early and mid-game play concentrate on achieving the Perfect score instead of just going out.

When playing Foo there are two ways to approach Perfect. In High Foo a player continues to only keep scores that result in a warp. Thus the only acceptable fourth level score is Perfect and great care must be taken earlier in the game to line up scores at lower levels to make the final jump to Perfect as easy as possible. In Low Foo a player may keep a score that gets them to Perfect even if it does not result in a warp. This allows players to get to anywhere on the fourth level then jump to Perfect.

As can be seen in Table 2, High Foo is quite challenging and results in an expected number of turns until Perfect far above the unconstrained approach. Low Foo is considerably easier, but still puts players at a significant disadvantage. Despite this, many players do play Foo variants as it is totally cool.

5.3 Wimp Marks

Many players do not like to keep wimp marks (scores that are 50 points away from a warp). To see how much this affects play, I made a pair of value functions that result in the minimum expected number of turns to going out and Perfect respectively without keeping any wimp marks along the way.

The results shown in Table 2 confirm that not taking wimp marks does not significantly affect the results of the strategy — people who take wimp marks really are just wimpy.

5.4 Aggressive

Another strategy to consider is a strategy that values the number of turns taken exponentially instead of linearly. This means the strategy will more aggressively try to go out earlier: increasing the total expected number of turns and the variance in order to increase the chance of a quick win. In previous multi-turn simulations, the expected value of a stopping state was recursively defined to be the expected value of the next starting state minus one; for the aggressive strategies the recursive value is multiplied by a fixed constant ($\lambda < 1$) instead. This results in optimizing for the maximum $\lambda^{\text{number of turns}}$.

Figure 9 shows a sampled distribution of turns until going out for a range of λ values, along with the linear value function. As $\lambda \Longrightarrow 1$, the aggressive strategy approaches the unconstrained linear strategy.

Aggressive strategies do have a higher chance of ending the game very early, but at a significant cost to their expected value and variance. Given normal end game dynamics, this does not seem a very profitable tradeoff.

5.5 Going Out With More Points

Rarely are players content to go out with only 5000 points. Typically players will pick a threshold score they deem acceptable to go out at, based on the position of the other players on the scoreboard, and only accept scores that exceed the threshold.

Using the same technique as above, it is possible to optimize the number of turns taken to get to a given threshold going out score. The value function for this strategy returns success only when the target score is reached, and returns a large negative value for scores that would put the player out but do not exceed the threshold.

Figure 10 shows the expected number of turns it takes to go out with at least a threshold number of points, starting from no score.

6 Last Rolls

Another critical phase of the Games is Last Rolls. Since Last Rolls involves a fixed and small number of turns, only one per player, it is much more computationally tractable than full-game strategies. In addition to increasing computability, splitting games into two phases — the main game and Last Rolls — is a useful and natural decomposition for reasoning about the Games.

Unlike all the simulations detailed above, which only involve a single player, Last Rolls is predominately about inter-player interactions. It is simply not possible to consider Last Rolls in the context of a single player. Not only will each player have slightly different objectives (some trying for additional warps, some trying for a Come From Behind), often those objectives will directly depend on other players' actions in Last Rolls. For example, if the last player to roll is sitting at the Perfect score, this may affect what score players who roll earlier decide to keep. Even the order of players matters; a player with the Perfect score rolling last is very different from a player with the Perfect score rolling first.

In order to make this problem tractable, I assume players have perfect knowledge of what opponents who roll after them will do. For example, if the second to last player knows that the last player will not accept anything other than a Come From Behind they can compute the probability that the opponent will succeed and factor that into their own strategy. Knowledge of players who rolled previously during Last Rolls is not required to determine an optimal strategy; players need only know the current score to beat, their score, the scores of all remaining opponents still to take their last rolls, and what each remaining opponent will do in each possible situation.

The very last player to roll does not have any remaining opponents, so it is easy to define a value function for the last player. The function need only take into account the score to beat and the player's current score. Once a value function is defined, the optimal strategy to maximize that value can easily be computed with an `OptStrategy`. Given that strategy, it is possible to compute the probability that the last player will keep a given score. This distribution of the probability of the player keeping each score (P_{keep}) is sufficient to fully characterize the strategy of the player, at least for the purposes of other players' Last Rolls calculations. Other players don't care about the roll-by-roll strategy; they only care about what the player will end up keeping.

With the output P_{keep} distribution from the optimal strategy for the final player to serve as an input to the value function, the optimal strategy for the second to last player can be computed in the same fashion. And with the second to last player's behavior known, this process can be repeated for the third to last player. This can be extended to an arbitrary number of players in the game, however the computational complexity increases by a factor of 100 for each additional player (there are 100 possible scores between 0 and 4950).

I computed the optimal strategy set for a number of combinations of opponents with simple value functions. The results comprise over a gigabyte of text, so I have only included a few sample points from the full output. These, along with a description of the various value functions used, are shown in Table 3.

Table 3: Probability of being Come On at various going out scores.

To Beat	Opponent Scores	Total P_{cfb}	To Beat	Opponent Scores	Total P_{cfb}
Worst Case: Each player values only beating the current top score, regardless of the odds later players getting the Come From Behind. Note that these results are identical to the results from §4.2			*King of the Hill*: Each player tries to optimize the chance of getting a Come From Behind (and keeping it even after remaining opponents roll).		
5000	4000,4000,4000	.406635313063	5000	4000,4000,4000	.40662825182
5000	4000,4000,4500	.542536693755	5000	4000,4000,4500	.542526350155
5000	4500,4000,4000	.542536693755	5000	4500,4000,4000	.54252257127
5000	4950,4950,4950	.999454681511	5000	4950,4950,4950	.987203365028
6000	4000,4000,4000	.0883834110222	6000	4000,4000,4000	.0883833170048
6000	4000,4000,4950	.192534548805	6000	4000,4000,4950	.192534382253
6000	4950,4000,4000	.192534548805	6000	4950,4000,4000	.192534548805
6000	4950,4950,4000	.284786539916	6000	4950,4950,4000	.284786539916
6000	4950,4950,4950	.366498848058	6000	4950,4950,4950	.36649368473
7000	4000,4000,4000	.0177513548539	7000	4000,4000,4000	.0177513548539
7000	4000,4000,4950	.039885991255	7000	4000,4000,4950	.039885991255
7000	4950,4000,4000	.039885991255	7000	4950,4000,4000	.039885991255
7000	4950,4950,4000	.0615218312148	7000	4950,4950,4000	.0615218312148
7000	4950,4950,4950	.0826701149401	7000	4950,4950,4950	.0826701149401
8000	4000,4000,4000	.00389765510113	8000	4000,4000,4000	.00389765510113
8000	4000,4000,4950	.0080740476387	8000	4000,4000,4950	.0080740476387
8000	4950,4000,4000	.0080740476387	8000	4950,4000,4000	.0080740476387
8000	4950,4950,4000	.0122329296717	8000	4950,4950,4000	.0122329296717
8000	4950,4950,4950	.0163743746171	8000	4950,4950,4950	.0163743746171
10000	4950,4950,4950	.00109158757115	10000	4950,4950,4950	.00109158757115
Warps Per Game: Each player tries to maximize their warps per game, counting both level warps and potential warps for getting the Come From Behind.			*Warps Per Game - CFB = 20*: Each player tries to maximize their level warps plus 20 warps for a Come From Behind. This attempts to model the social value many players place on getting a CFB.		
5000	4000,4000,4000	.406627545688	5000	4000,4000,4000	.40662825182
5000	4000,4000,4500	.542453397228	5000	4000,4000,4500	.542526350155
5000	4500,4000,4000	.542522026857	5000	4500,4000,4000	.54252257127
5000	4950,4950,4950	.994655683191	5000	4950,4950,4950	.98911083943
6000	4000,4000,4000	.0671649417401	6000	4000,4000,4000	.0883833170048
6000	4000,4000,4950	.127667248375	6000	4000,4000,4950	.190482325075
6000	4950,4000,4000	.111814067841	6000	4950,4000,4000	.192534548805
6000	4950,4950,4000	.154386658002	6000	4950,4950,4000	.284786539916
6000	4950,4950,4950	.207897097066	6000	4950,4950,4950	.36649368473
7000	4000,4000,4000	.00602577599333	7000	4000,4000,4000	.00911487525615
7000	4000,4000,4950	.0119975651047	7000	4000,4000,4950	.0178194063962
7000	4950,4000,4000	.0119975651047	7000	4950,4000,4000	.0178194063962
7000	4950,4950,4000	.0179334757553	7000	4950,4950,4000	.0264474716972
7000	4950,4950,4950	.0238337235026	7000	4950,4950,4950	.0349997428812
8000	4000,4000,4000	.00101606767444	8000	4000,4000,4000	.0012806599141
8000	4000,4000,4950	.00181544173678	8000	4000,4000,4950	.00213817800584
8000	4950,4000,4000	.00181544173678	8000	4950,4000,4000	.00213817800584
8000	4950,4950,4000	.0026141761503	8000	4950,4950,4000	.00299495981738
8000	4950,4950,4950	.00341227142684	8000	4950,4950,4950	.0038510059809
10000	4950,4950,4950	.000396496585273	10000	4950,4950,4950	.000413301047671

7 Implementation Details — The October Framework

The simulation and computation framework used in this study is written in stock C++. It should be portable to just about any standard C++ compiler that includes TR1, although all testing was done with `g++`. Boost may substitute for TR1 on older compilers. A previous version of the framework was written in Common Lisp, and is used to generate lookup tables to increase performance of the C++ version.

The basic data structures in the framework are `Frisbee`, `Scoreboard`, and `Strategy`. A `Frisbee` holds a score and a number of live dice while a `Scoreboard` holds the scores of all players in the game. `Strategy` is a virtual interface with one method: given a `Frisbee`, a `Scoreboard`, and a roll, return which dice — if any — to reroll.

The most significant piece of the framework is `OptStrategy`: a `Strategy` that takes a value function, recursively walks the probability tree, and returns the action with the highest expected value. The value function is defined over `Frisbee` and `Scoreboard` objects, and represents "what is the value of stopping here?". By picking a value function, such as the stopping score, the class can be used to find the optimal strategy for any number of objectives. The recursive search is bounded by the `MAX_RECURSE_DEPTH` variable, which cuts the tree at a fixed "improbability depth", as described in §7.1.

`OptStrategy` objects also take a hash function that maps from `Frisbee` and `Scoreboard` to hash keys. This allows the `OptStrategy` to memoize results as it descends the tree and saves extensive computation. The hash function for an `OptStrategy` should return a unique key for each state that should be treated differently by the optimizer. Different value functions may use different hash functions that only encode subsets of game state in order to save space and repeated computation. For example, most basic strategies do not need to store the relative scores of opponents in their hash, whereas going out and last roll strategies do need such information.

Hashed decision tables from the `OptStrategy` classes may be checkpointed to text files. This allows computation to be saved between runs of the program.

7.1 Bounding Recursive Descent

Since the tree of potential game play is infinitely deep, it is necessary to cut off computation at some point when calculating expected value of an action. The most natural way to cut off is to use the probability of reaching the node in question, not the flat recursion depth. I call this the *improbability depth* of the node since the magnitude of the probability of reaching the node becomes smaller the deeper the search gets.

To implement the cutoff, `OptStrategy` considers strategies to to have stopped rolling and kept their score once the threshold improbability has been exceeded. This has a minimal impact on the expected value computation and resulting action, since the contribution is multiplied by the probability.

Because `OptStrategy` memoizes its results, a decision from deep in the tree which may not be optimal because of the cutoff might be reused when the result is now more important to the expected value computation. To counter this, the probability of the node at the time of the previous computation is stored along with the result and the result is recomputed when needed. The run-time parameter `RECOMPUTE_FACTOR` controls when a node will be recomputed.

To save computation across runs with different `MAX_RECURSE_DEPTH` cutoffs the saved improbability is expressed in terms relative to the cutoff. By using this *improbability remaining* metric, memoized results stay meaningful even when the cutoff changes. When the cutoff increases some nodes may need to be recomputed, but typically some results may be reused.

By noting when recomputation changes a cached result it is possible to observe the convergence properties of a strategy decision table. If a strategy does not change with successive increases in cutoff depth, it can be marked as fully converged by setting the stored probability associated with each decision to infinite improbability remaining. This causes `OptStrategy` to never recompute the decisions.

7.2 Parallelizing

Although single-player simulations are computationally feasible on single machines, multi-player simulations require the ability to scale CPU power horizontally. The Last Rolls optimizations were primarily conducted on Amazon's Elastic Compute Cloud (EC2).

By splitting the computation into single turn units, each with its own `OptStrategy` invocation, the bulk of the computation is "embarrassingly parallel" — each potential scoreboard can be computed independently.

The Hadoop framework provides an excellent platform for parallel computation using the MapReduce algorithm. All the Last Rolls simulation code uses simple Python scripts to glue the C++ `OptStrategy` implementation into the Hadoop "Streaming" interface. Using Hadoop makes computing in Amazon's compute cloud very easy and effective.

8 Future Work

I also wrote a facility to play strategies against each other in full games using the PRNG to generate rolls. This allows generation of game-end statistics for a given `Strategy` set, however I have not had time to seriously investigate results from such simulations.

For simulations that investigate behavior over entire games — as opposed to only single turns — I created a `CombinerStrategy` class which takes an ordered list of `pair<Predicate, Strategy>`. Predicates are defined over the same space as hash functions. The strategy iterates over the list, and uses the first sub-strategy whose predicate returns true. Again, I have not had time to investigate results from this approach.

With the October framework designed and many basic strategy building blocks already implemented, it is possible to build quite complex and realistic strategies. The limiting factors are only CPU time, RAM, and imagination.

Some future work items that build upon the current framework include:

- Using the results from Last Rolls, optimize the main phase of the game. Since the search space is so large, this may require a different approach (I've been considering simulated annealing or parallel tempering).
- Compose strategies modeling the whole game using `CombinerStrategy` and determine how they perform in repeated play.
- Compare basic composed strategies against each other. Since strategy performance is heavily influenced by the other players in the game, this could be very intensive.

I'd like to thank all the people who have helped me with my work on the Games over last several years. You know who you are.

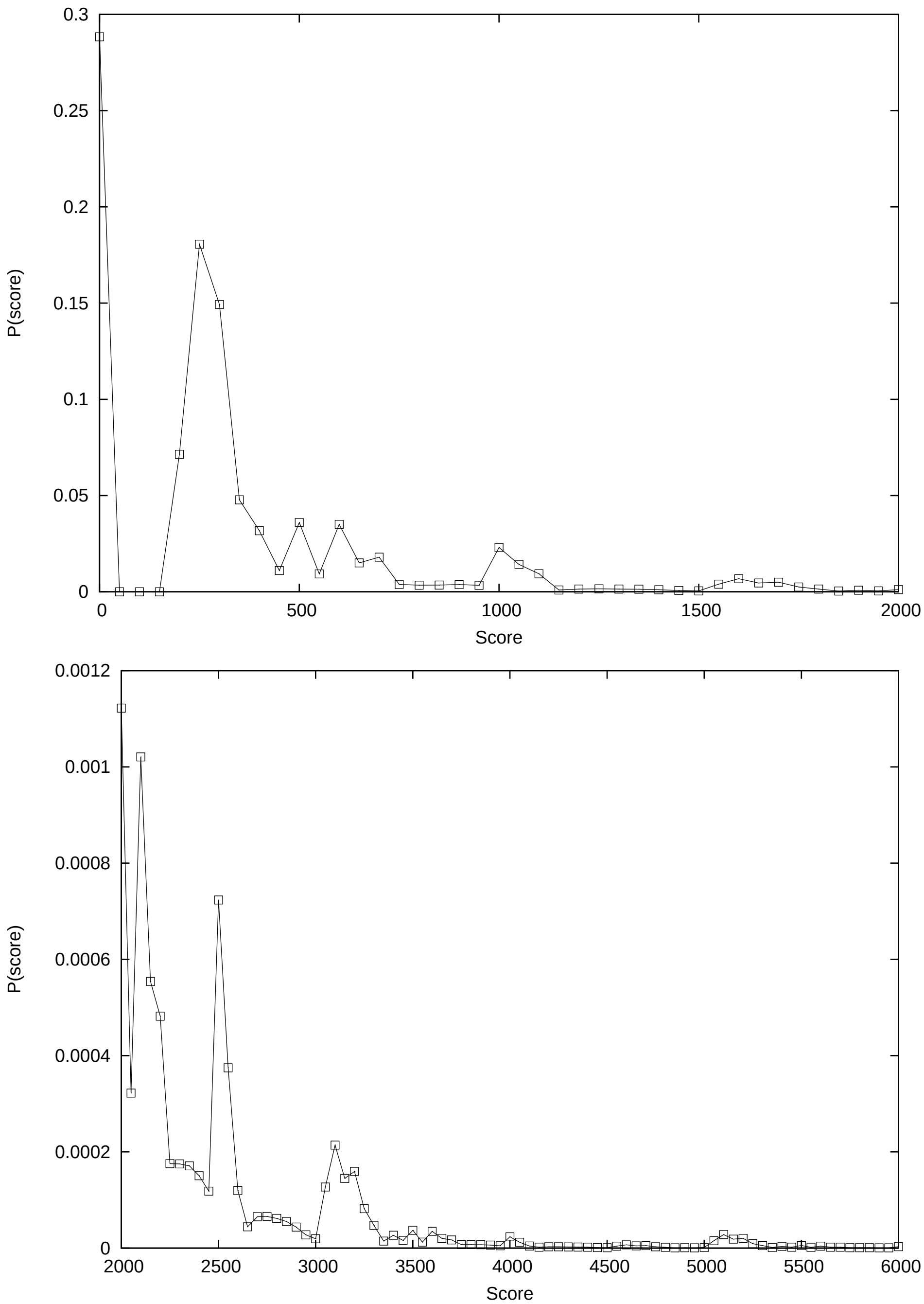

Figure 1: Probability distribution of scores (P_{keep}) when rolling for maximum points per turn. Note multiple scales.

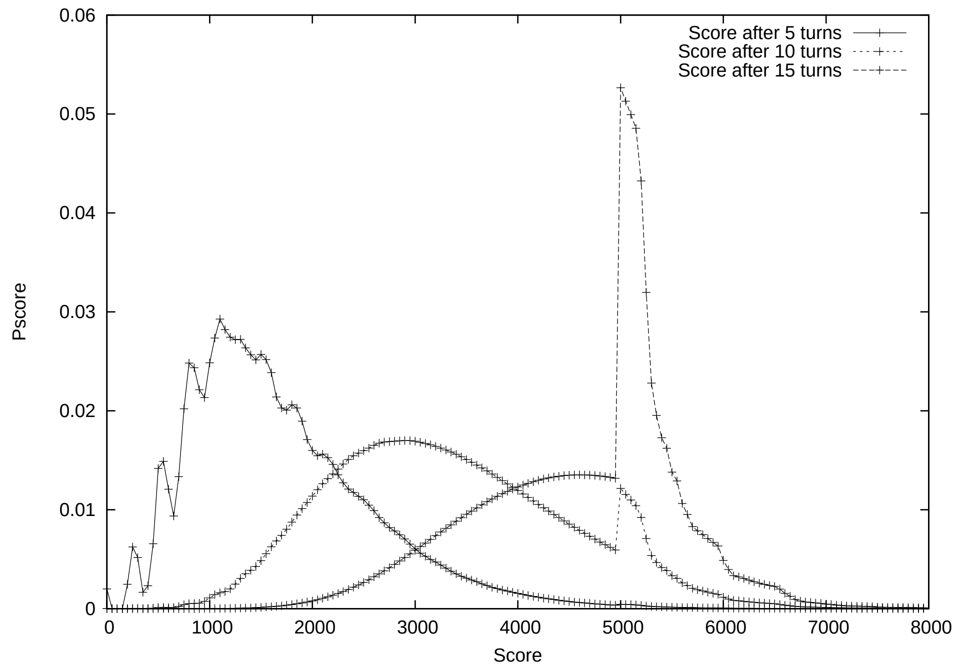

Figure 2: Probability distribution of score totals after N turns.

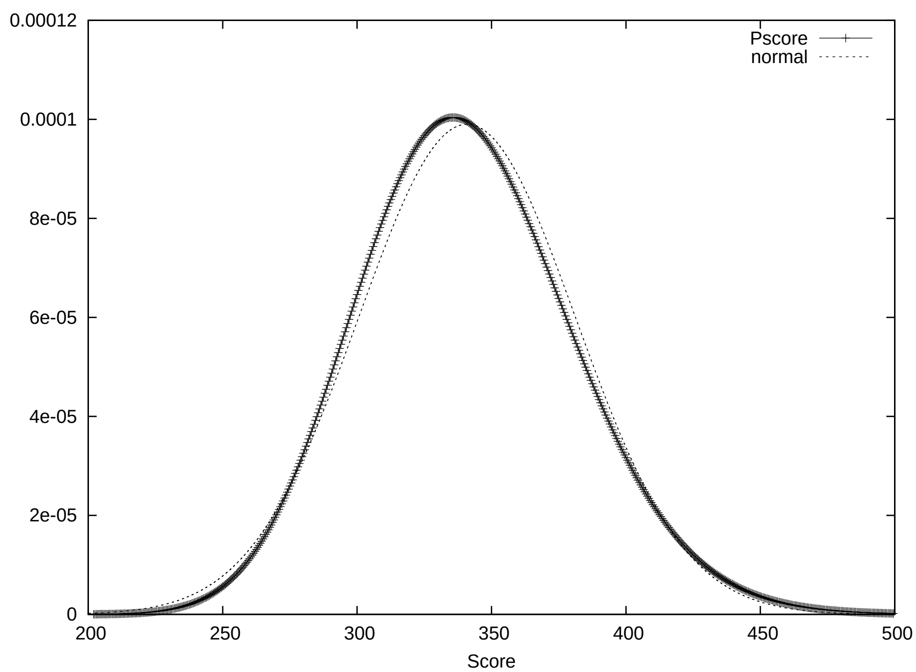

Figure 3: Probability distribution of points per turn in 100 turns.

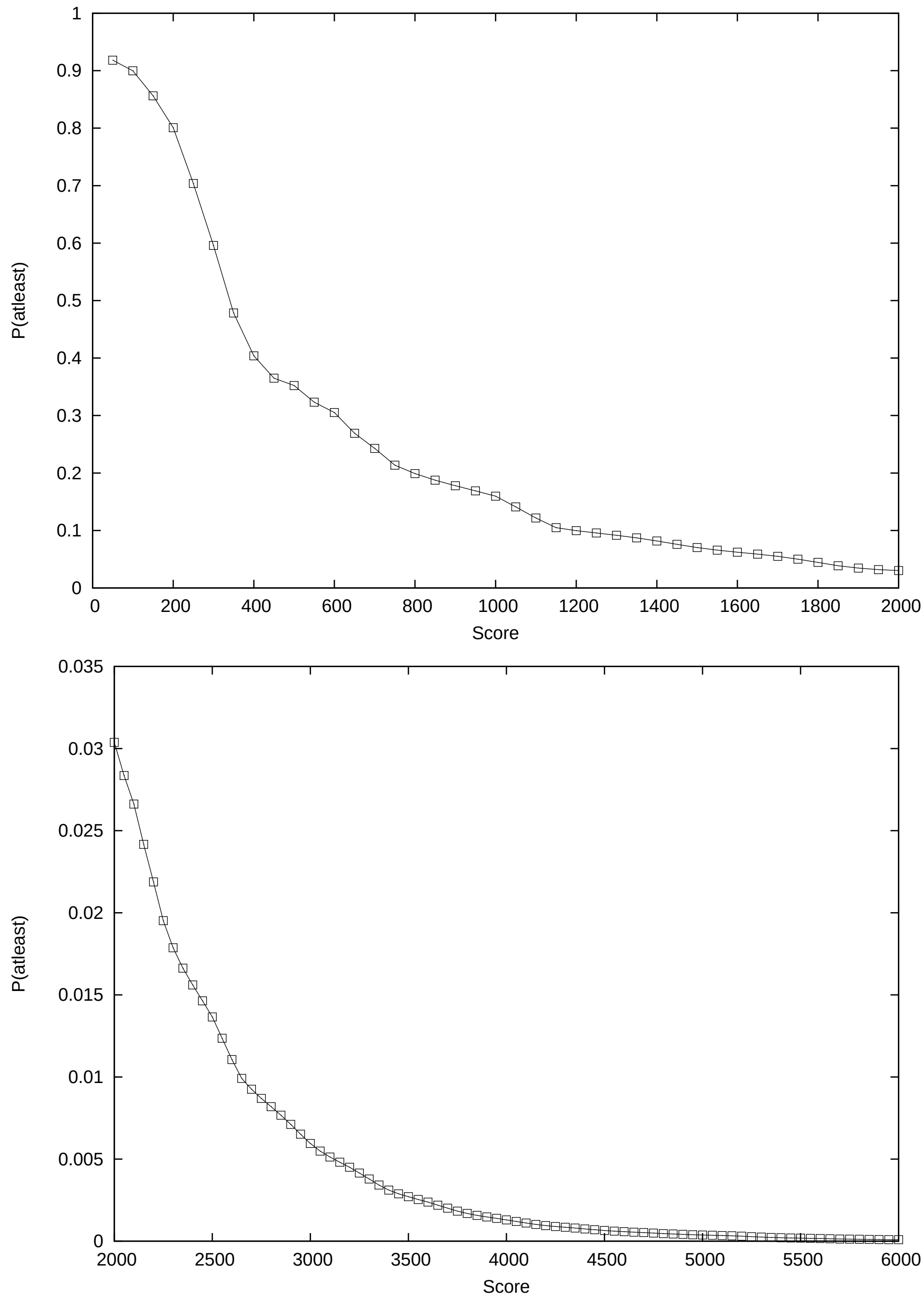

Figure 4: P_{atleast}: Probability of reaching at least N points in one turn. Note multiple scales.

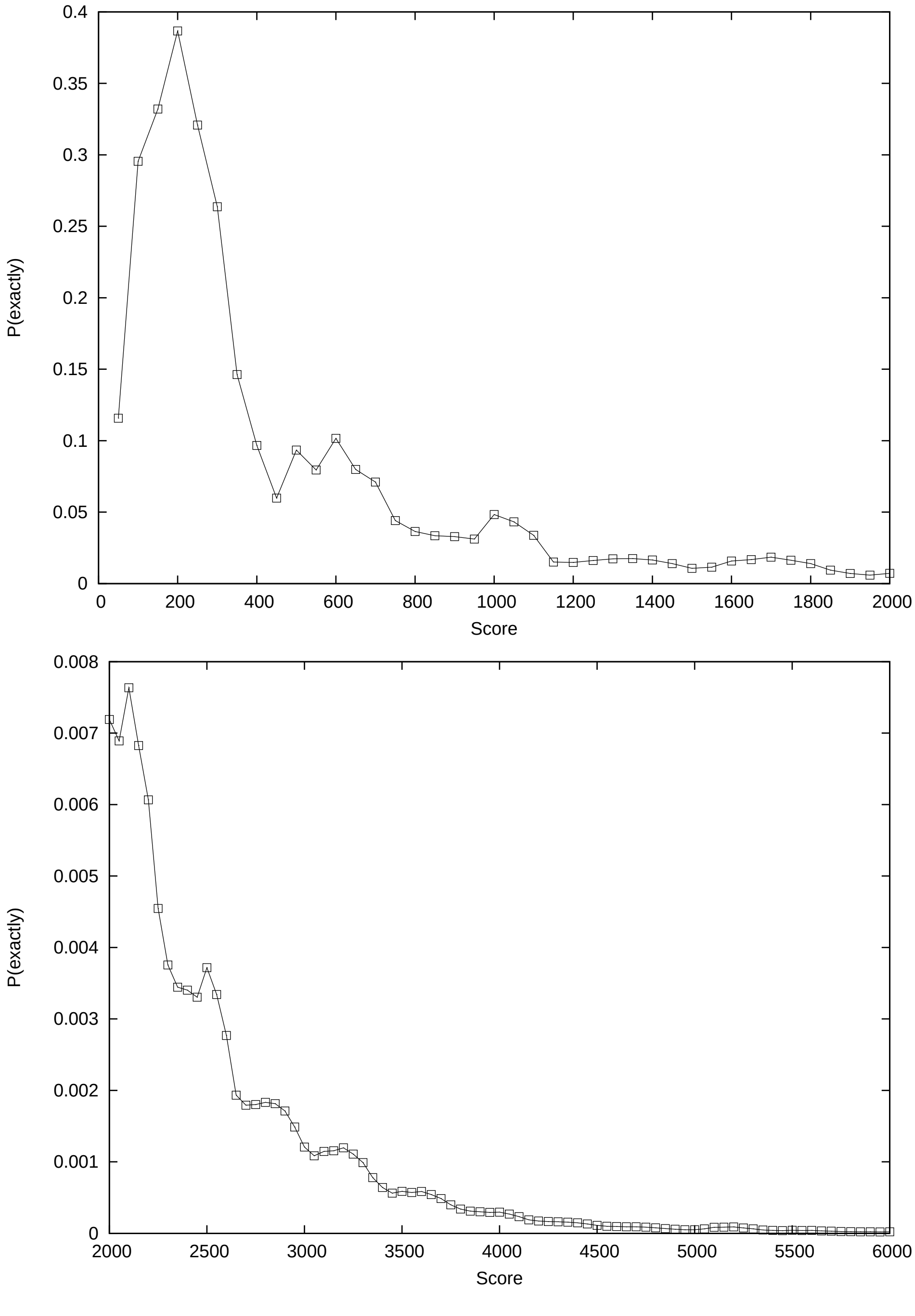

Figure 5: P_{exactly}: Probability of hitting exactly N points in one turn. Note multiple scales.

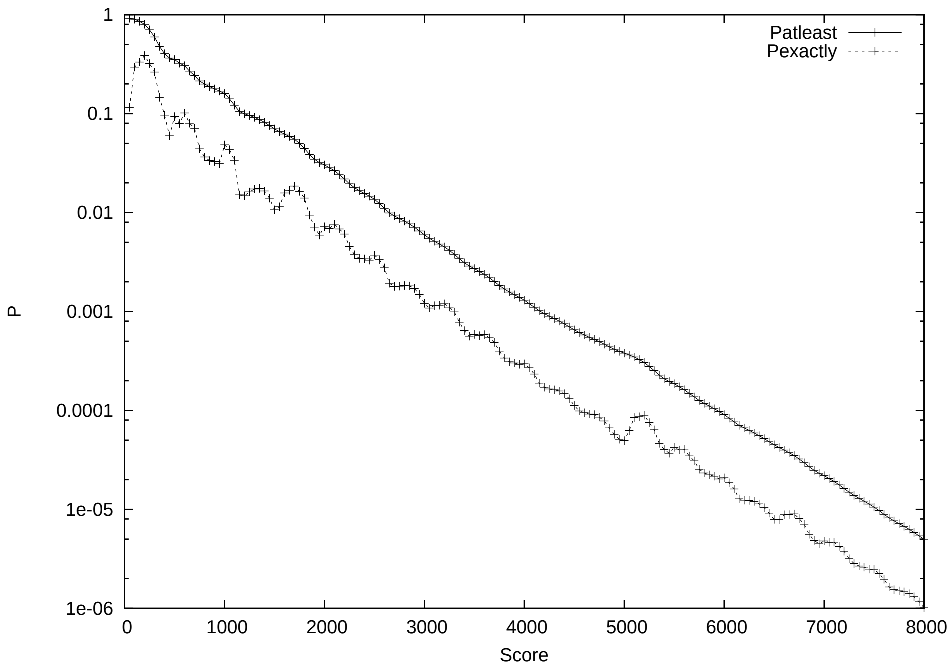

Figure 6: $\log_{10}(P_{\text{atleast}})$ and $\log_{10}(P_{\text{exactly}})$

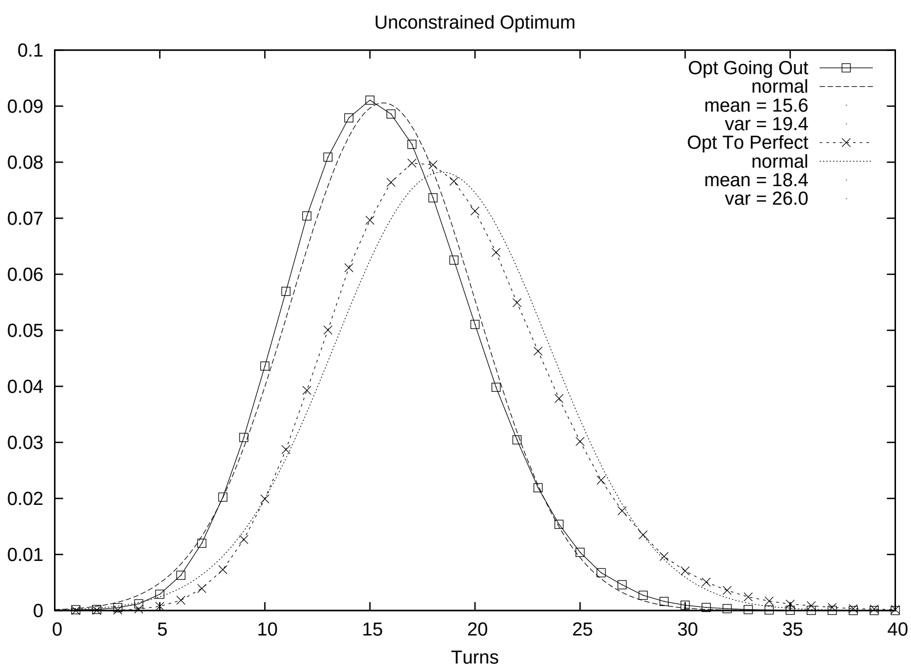

Figure 7: Distribution of number of turns until going out and Perfect for unconstrained optimization.

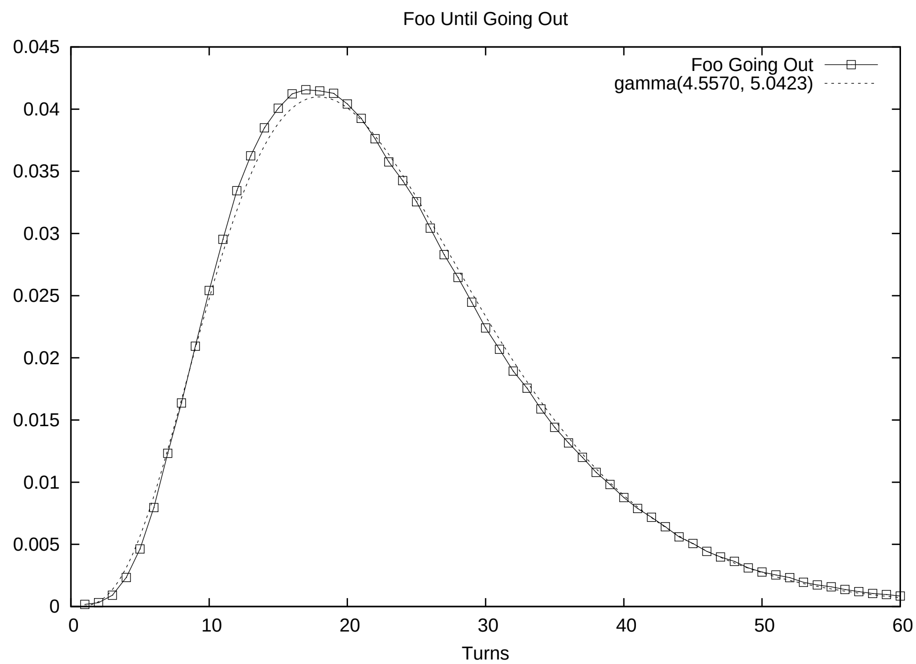

Figure 8: Distribution of number of turns until going out when playing Foo, along with best fit Gamma distribution.

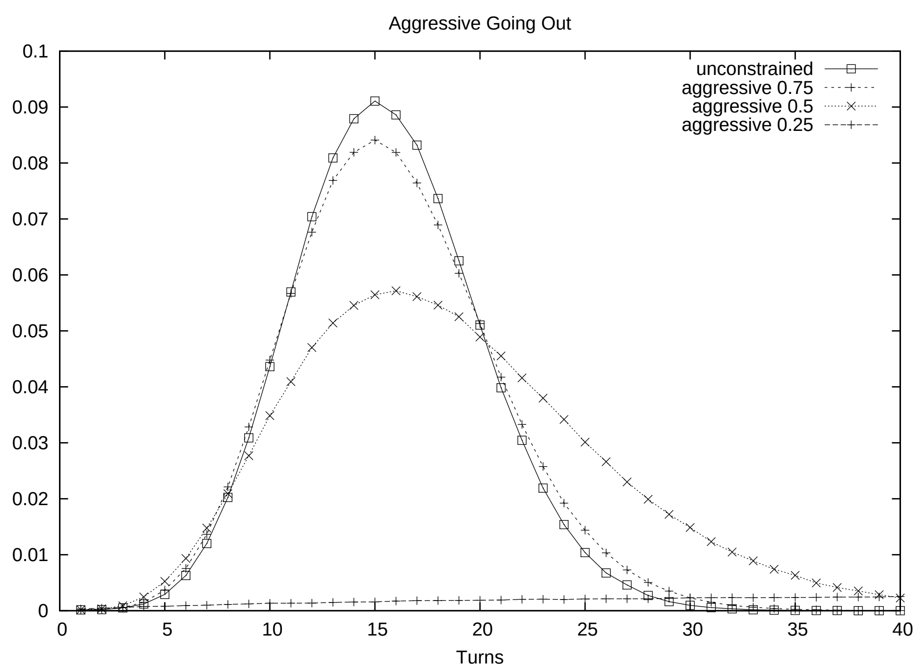

Figure 9: Distribution of number of turns until going out for exponentially aggressive strategies.

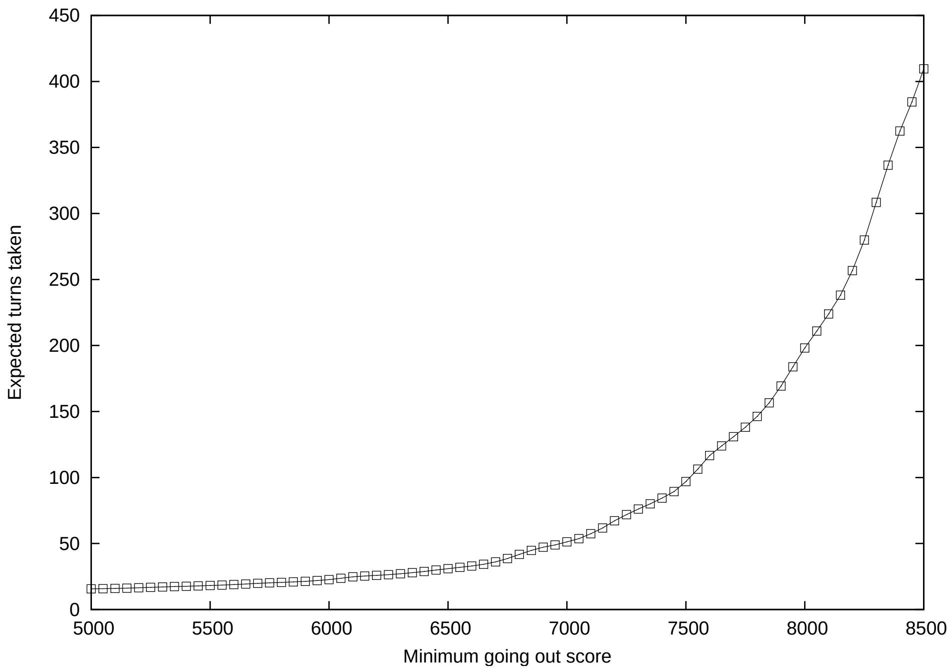

Figure 10: Expected number of turns taken to go out with at least a threshold number of points.

A Maximum Points Per Roll Strategy Table

roll	held	reroll
16	150--	stop
56	150--	stop
116	100--	stop
156	100--	stop
166	100--	stop
556	100--	stop
566	100	12
566	150--	stop
1116	50--	stop
1156	50--	stop
1166	50--	stop
1466	50--100	123
1466	300--	stop
1556	50--	stop
1566	50	123
1566	100--	stop
2226	50--	stop
3336	50--	stop
4446	50--	stop
4566	50--300	023
4566	350--	stop
5556	50--	stop
5566	50--100	123
5566	300--	stop
6666	50--	stop
11116	0--	stop
11156	0--	stop
11166	0--	stop
11466	0	1234
11466	250--	stop
11556	0--	stop
11566	0	1234
11566	250--	stop
12226	0--	stop
13336	0--	stop
14446	0--	stop
14466	0--800	1234
14466	850--	stop
14566	0--500	1234
14566	550--	stop
15556	0--	stop
15566	0	1234
15566	250--	stop
16666	0--	stop
22256	0--	stop
22266	0--	stop
33356	0--	stop
33366	0--	stop
44456	0--	stop
44466	0--	stop
44566	0--850	0134
44566	900--	stop
45566	0--550	0234
45566	600--	stop
55556	0--	stop
55566	0--	stop
56666	0--	stop
66666	0--	stop

B Atleast-N Points Strategy Table

roll	to-go	reroll
16	0--100	stop
16	150--	1
56	0--50	stop
56	100--	1
116	0--200	stop
116	250--	2
156	0--150	stop
156	200	12
156	250--	2
166	0--100	stop
166	150--	12
556	0--100	stop
556	150	12
556	200--	2
566	0--50	stop
566	100--	12
1116	0--1000	stop
1116	1050--	3
1156	0--250	stop
1156	300	23
1156	350--	3
1166	0--200	stop
1166	250--750	23
1166	800	123
1166	850--1050	23
1166	1100--1300	123
1166	1350--1500	23
1166	1550--1600	123
1166	1650--	23
1466	0--100	stop
1466	150--	123
1556	0--200	stop
1556	250--300	123
1556	350--	3
1566	0--150	stop
1566	200--300	123
1566	350--500	23
1566	550--	123
2226	0--200	stop
2226	250--	3
3336	0--300	stop
3336	350--	3
4446	0--400	stop
4446	450--	3
4566	0--50	stop
4566	100--	023
5556	0--500	stop
5556	550--	3
5566	0--100	stop
5566	150--250	123
5566	300--450	23
5566	500--	123
6666	0--600	stop
6666	650--	3
11116	0--1100	stop
11116	1150--	4
11156	0--1050	stop
11156	1100	34
11156	1150--	4
11166	0--1000	stop
11166	1050--	34
11466	0--200	stop
11466	250--350	234
11466	400--	1234
11556	0--300	stop
11556	350	234
11556	400	1234
11556	450--650	4
11556	700--1100	1234
11556	1150--1450	4
11556	1500	1234
11556	1550--	4
11566	0--250	stop
11566	300--350	234
11566	400--	1234
12226	0--300	stop
12226	350--400	1234
12226	450--650	4
12226	700--1100	1234
12226	1150--1450	4
12226	1500	1234
12226	1550--	4
13336	0--400	stop
13336	450--	4
14446	0--500	stop
14446	550--	4
14466	0--100	stop
14466	150--	1234
14566	0--150	stop
14566	200--	1234
15556	0--600	stop
15556	650--	4
15566	0--200	stop
15566	250--	1234
16666	0--700	stop
16666	750--	4
22256	0--250	stop
22256	300--350	0124
22256	400--600	4
22256	650--1050	0124
22256	1100--1400	4
22256	1450	0124
22256	1500--	4
22266	0--200	stop
22266	250--	34
33356	0--350	stop
33356	400	34
33356	450--	4
33366	0--300	stop
33366	350--	34
44456	0--450	stop
44456	500	34
44456	550--	4
44466	0--400	stop
44466	450--	34
44566	0--50	stop
44566	100--	0134
45566	0--100	stop
45566	150--	0234
55556	0--550	stop
55556	600	34
55556	650--	4
55566	0--500	stop
55566	550--	34
56666	0--650	stop
56666	700	04
56666	750--	4
66666	0--600	stop
66666	650--	34

C Exactly-N Points Strategy Table

roll	to-go	reroll
16	0--150	stop
16	200--	1
56	0--100	stop
56	150--	1
116	0--100	stop
116	150	12
116	200	stop
116	250--300	12
116	350--	2
156	0--50	stop
156	100	02
156	150	stop
156	200	12
156	250	02
156	300--550	2
156	600	12
156	650--1300	2
156	1350--1400	12
156	1450--	2
166	0--100	stop
166	150--	12
556	0--100	stop
556	150	12
556	200--500	2
556	550	12
556	600--1250	2
556	1300--1350	12
556	1400--	2
566	0--50	stop
566	100--	12
1116	0--100	stop
1116	150--250	123
1116	300	23
1116	350--400	123
1116	450--750	23
1116	800--900	123
1116	950	23
1116	1000	stop
1116	1050	23
1116	1100--	3
1156	0--50	stop
1156	100--150	013
1156	200	123
1156	250	stop
1156	300	23
1156	350	013
1156	400--650	3
1156	700	23
1156	750--1200	3
1156	1250	013
1156	1300--1400	3
1156	1450--1500	23
1156	1550--	3
1166	0--100	stop
1166	150	123
1166	200	stop
1166	250	123
1166	300	23
1166	350--400	123
1166	450--750	23
1166	800--900	123
1166	950--1150	23
1166	1200--1350	123
1166	1400--1500	23
1166	1550--1700	123
1166	1750--1800	23
1166	1850--1950	123
1166	2000--2250	23
1166	2300--2400	123
1166	2450--2650	23
1166	2700--2850	123
1166	2900--3000	23
1166	3050--3200	123
1166	3250--3350	23
1166	3400--3500	123
1166	3550--3750	23
1166	3800--3900	123
1166	3950--4150	23
1166	4200--4300	123
1166	4350--4550	23
1166	4600--4650	123
1166	4700--4900	23
1166	4950--5000	123
1166	5050--5250	23
1166	5300--5400	123
1166	5450--5750	23
1166	5800--5850	123
1166	5900--6150	23
1166	6200--6300	123
1166	6350--6500	23
1166	6550--6600	123
1166	6650--6800	23
1166	6850--6950	123
1166	7000--7250	23
1166	7300--7350	123
1166	7400--7650	23
1166	7700--7800	123
1166	7850--8050	23
1166	8100	123
1166	8150--10300	23
1166	10350--10400	123
1166	10450--	23
1466	0--100	stop
1466	150--	123
1556	0--50	stop
1556	100--150	023
1556	200	stop
1556	250--300	123
1556	350--600	3
1556	650--700	123
1556	750--1000	3
1556	1050	123
1556	1100--1350	3
1556	1400--1450	123
1556	1500--1700	3
1556	1750	123
1556	1800--2100	3
1556	2150--2200	123
1556	2250--2850	3
1556	2900--2950	123
1556	3000--	3
1566	0--50	stop
1566	100	023
1566	150	stop
1566	200--300	123
1566	350	023
1566	400--550	23
1566	600--750	123
1566	800--850	023
1566	900--1100	123
1566	1150	023
1566	1200	123
1566	1250--1300	023
1566	1350--1500	123
1566	1550	023
1566	1600	123
1566	1650	023
1566	1700--1800	123
1566	1850--1900	023
1566	1950--2050	23
1566	2100--2250	123
1566	2300--2350	023
1566	2400--2700	123
1566	2750	023
1566	2800--3000	123

1566	3050	023
1566	3100--3350	123
1566	3400--3450	023
1566	3500	23
1566	3550--5250	123
1566	5300--5350	023
1566	5400--5500	23
1566	5550--5700	123
1566	5750--5800	023
1566	5850	23
1566	5900--6800	123
1566	6850--6900	023
1566	6950--7000	23
1566	7050--	123
2226	0--250	stop
2226	300--	3
3336	0--350	stop
3336	400--	3
4446	0--450	stop
4446	500--	3
4566	0--50	stop
4566	100--	023
5556	0--50	stop
5556	100--250	123
5556	300--450	23
5556	500	stop
5556	550--600	123
5556	650--5400	3
5556	5450	23
5556	5500--5550	123
5556	5600--	3
5566	0--100	stop
5566	150--250	123
5566	300--500	23
5566	550--750	123
5566	800	23
5566	850--1750	123
5566	1800--2000	23
5566	2050--2250	123
5566	2300	23
5566	2350--3350	123
5566	3400--3450	23
5566	3500--5250	123
5566	5300--5450	23
5566	5500--5750	123
5566	5800	23
5566	5850--6800	123
5566	6850--6950	23
5566	7000--	123
6666	0--650	stop
6666	700--	3
11116	0--100	stop
11116	150--250	1234
11116	300--400	234
11116	450--600	1234
11116	650	234
11116	700	1234
11116	750--800	234
11116	850--1000	1234
11116	1050	34
11116	1100	stop
11116	1150	234
11116	1200	34
11116	1250--	4
11156	0--50	stop
11156	100--150	0124
11156	200	1234
11156	250	124
11156	300--400	234
11156	450--500	0124
11156	550--650	24
11156	700	1234
11156	750--800	234
11156	850--1000	1234
11156	1050	stop
11156	1100	34
11156	1150--1450	4
11156	1500	34
11156	1550--2200	4
11156	2250--2300	34
11156	2350--	4
11166	0--100	stop
11166	150--250	1234
11166	300--400	234
11166	450--600	1234
11166	650	234
11166	700	1234
11166	750--800	234
11166	850--950	1234
11166	1000	stop
11166	1050--1100	34
11166	1150	234
11166	1200--	34
11466	0--100	stop
11466	150	1234
11466	200	stop
11466	250	1234
11466	300--400	234
11466	450--600	1234
11466	650	234
11466	700	1234
11466	750--800	234
11466	850--1100	1234
11466	1150	234
11466	1200--2150	1234
11466	2200	234
11466	2250--5550	1234
11466	5600--5700	234
11466	5750--	1234
11556	0--50	stop
11556	100--150	0134
11556	200	1234
11556	250	134
11556	300	stop
11556	350--400	234
11556	450--650	4
11556	700	1234
11556	750--800	234
11556	850	1234
11556	900--1000	4
11556	1050	0134
11556	1100	1234
11556	1150	234
11556	1200--1350	0134
11556	1400	4
11556	1450--1600	1234
11556	1650--1750	4
11556	1800--1900	1234
11556	1950--2200	4
11556	2250--2300	1234
11556	2350--2600	4
11556	2650	1234
11556	2700--2950	4
11556	3000--3050	1234
11556	3100--3300	4
11556	3350--3400	1234
11556	3450--3700	4
11556	3750--3800	1234
11556	3850--4500	4
11556	4550	1234
11556	4600--	4
11566	0--50	stop
11566	100--150	0134
11566	200	1234
11566	250	stop
11566	300--400	234
11566	450--500	0134
11566	550--650	34
11566	700	1234
11566	750--800	234
11566	850--1000	1234
11566	1050	0134
11566	1100	1234

11566	1150	234
11566	1200--1350	0134
11566	1400--1600	1234
11566	1650--1700	0134
11566	1750--1900	1234
11566	1950--2000	0134
11566	2050--2100	34
11566	2150	1234
11566	2200	234
11566	2250--2350	1234
11566	2400	0134
11566	2450--5300	1234
11566	5350--5500	0134
11566	5550--5600	34
11566	5650--5700	234
11566	5750--6900	1234
11566	6950--7000	0134
11566	7050--	1234
12226	0--100	stop
12226	150--250	1234
12226	300	stop
12226	350--400	1234
12226	450--650	4
12226	700--850	1234
12226	900--1050	4
12226	1100--1350	1234
12226	1400	4
12226	1450--1600	1234
12226	1650--1750	4
12226	1800--1900	1234
12226	1950--2200	4
12226	2250--2300	1234
12226	2350--2600	4
12226	2650	1234
12226	2700--2950	4
12226	3000--3050	1234
12226	3100--3300	4
12226	3350--3400	1234
12226	3450--3700	4
12226	3750--3800	1234
12226	3850--4500	4
12226	4550	1234
12226	4600--	4
13336	0--100	stop
13336	150--300	1234
13336	350	04
13336	400	stop
13336	450--550	1234
13336	600--850	4
13336	900--950	1234
13336	1000--1050	4

13336	1100	1234
13336	1150--1250	4
13336	1300--1400	1234
13336	1450--1600	4
13336	1650--1700	1234
13336	1750--1950	4
13336	2000	1234
13336	2050--5400	4
13336	5450--5500	1234
13336	5550--	4
14446	0--100	stop
14446	150--400	1234
14446	450	04
14446	500	stop
14446	550--650	1234
14446	700--1050	4
14446	1100	1234
14446	1150--1300	4
14446	1350--1450	1234
14446	1500--1700	4
14446	1750--1800	1234
14446	1850--2000	4
14446	2050--2100	1234
14446	2150--5450	4
14446	5500--5600	1234
14446	5650--	4
14466	0--100	stop
14466	150--	1234
14566	0--50	stop
14566	100	0134
14566	150	stop
14566	200	1234
14566	250	134
14566	300--400	1234
14566	450--550	0134
14566	600--1000	1234
14566	1050	0134
14566	1100--1150	1234
14566	1200--1350	0134
14566	1400--1600	1234
14566	1650--1700	0134
14566	1750--1900	1234
14566	1950--2050	0134
14566	2100--2350	1234
14566	2400	0134
14566	2450--5300	1234
14566	5350--5500	0134
14566	5550--6900	1234
14566	6950--7000	0134
14566	7050--	1234
15556	0--50	stop

15556	100--150	0234
15556	200	1234
15556	250	234
15556	300--400	1234
15556	450--500	34
15556	550	04
15556	600	stop
15556	650--700	1234
15556	750--1050	4
15556	1100	1234
15556	1150--2100	4
15556	2150--2200	1234
15556	2250--5500	4
15556	5550	34
15556	5600--5650	1234
15556	5700--	4
15566	0--50	stop
15566	100--150	0234
15566	200	stop
15566	250	234
15566	300--400	1234
15566	450--550	34
15566	600--1000	1234
15566	1050	0234
15566	1100--1150	1234
15566	1200--1350	0234
15566	1400--1600	1234
15566	1650--1700	0234
15566	1750--1900	1234
15566	1950	034
15566	2000--2050	34
15566	2100--2350	1234
15566	2400	0234
15566	2450--5300	1234
15566	5350	0234
15566	5400	034
15566	5450--5550	34
15566	5600--6900	1234
15566	6950--7000	0234
15566	7050--	1234
16666	0--100	stop
16666	150--600	1234
16666	650	04
16666	700	stop
16666	750	1234
16666	800	04
16666	850--	4
22256	0--50	stop
22256	100--200	0124
22256	250	stop
22256	300	34

22256	350	0124
22256	400--600	4
22256	650--800	0124
22256	850--1000	4
22256	1050--1300	0124
22256	1350	4
22256	1400--1550	0124
22256	1600--1700	4
22256	1750--1850	0124
22256	1900--2150	4
22256	2200--2250	0124
22256	2300--2550	4
22256	2600	0124
22256	2650--2900	4
22256	2950--3000	0124
22256	3050--3250	4
22256	3300--3350	0124
22256	3400--3650	4
22256	3700--3750	0124
22256	3800--4450	4
22256	4500	0124
22256	4550--	4
22266	0--200	stop
22266	250--	34
33356	0--50	stop
33356	100--300	0124
33356	350	stop
33356	400	34
33356	450--500	0124
33356	550--750	4
33356	800	34
33356	850--900	0124
33356	950--1000	4
33356	1050	0124
33356	1100--1200	4
33356	1250--1350	0124
33356	1400--1500	4
33356	1550	34
33356	1600--1650	0124
33356	1700--1900	4
33356	1950	0124
33356	2000--5350	4
33356	5400--5450	0124
33356	5500--	4
33366	0--300	stop
33366	350--	34
44456	0--50	stop
44456	100--400	0124
44456	450	stop
44456	500	34
44456	550--600	0124
44456	650--850	4
44456	900	34
44456	950--1000	4
44456	1050	0124
44456	1100--1250	4
44456	1300--1400	0124
44456	1450--1600	4
44456	1650	34
44456	1700--1750	0124
44456	1800--1950	4
44456	2000--2050	0124
44456	2100--5400	4
44456	5450--5550	0124
44456	5600--	4
44466	0--400	stop
44466	450--	34
44566	0--50	stop
44566	100--	0134
45566	0--100	stop
45566	150	0234
45566	200	034
45566	250--350	0234
45566	400--500	034
45566	550--1900	0234
45566	1950	034
45566	2000--5350	0234
45566	5400--5450	034
45566	5500--	0234
55556	0--50	stop
55556	100--150	1234
55556	200	234
55556	250--350	1234
55556	400--500	234
55556	550	stop
55556	600	34
55556	650	1234
55556	700--950	4
55556	1000	34
55556	1050	1234
55556	1100--1700	4
55556	1750--1800	34
55556	1850--2050	4
55556	2100--2150	1234
55556	2200--5450	4
55556	5500--5600	1234
55556	5650--	4
55566	0--50	stop
55566	100--150	1234
55566	200	234
55566	250--350	1234
55566	400--450	234
55566	500	stop
55566	550--600	34
55566	650--700	1234
55566	750--1000	34
55566	1050	1234
55566	1100--1350	34
55566	1400--1450	1234
55566	1500--2000	34
55566	2050--2150	1234
55566	2200--5400	34
55566	5450	234
55566	5500--5650	1234
55566	5700--	34
56666	0--50	stop
56666	100--600	1234
56666	650	stop
56666	700	04
56666	750	1234
56666	800--1050	4
56666	1100	04
56666	1150--1800	4
56666	1850--1900	04
56666	1950--	4
66666	0--600	stop
66666	650--	34

D Atleast-N Points Probability Table

delta	probability
50	0.9183007
100	0.899732
150	0.8562233
200	0.8006625
250	0.7036211
300	0.5959953
350	0.4783706
400	0.4039132
450	0.3650821
500	0.3521494
550	0.3231298
600	0.3051504
650	0.2691242
700	0.2429048
750	0.2135996
800	0.1989502
850	0.1875024
900	0.1779815
950	0.1688911
1000	0.1596885
1050	0.1411531
1100	0.1216537
1150	0.1048424
1200	0.09981525
1250	0.09562498
1300	0.09155415
1350	0.08705305
1400	0.08180564
1450	0.07586563
1500	0.07019512
1550	0.0657659
1600	0.06226859
1650	0.05877344
1700	0.05507221
1750	0.05001743
1800	0.04451256
1850	0.0387322
1900	0.03459111
1950	0.03195024
2000	0.03037439
2050	0.02835302
2100	0.02661741
2150	0.02417058
2200	0.0218776
2250	0.01951944
2300	0.01786839
2350	0.01662351
2400	0.0156049
2450	0.01463702
2500	0.01365408
2550	0.01236217
2600	0.01105964
2650	0.009917349
2700	0.009250047
2750	0.008695188
2800	0.008188426
2850	0.007672316
2900	0.007114773
2950	0.00651861
3000	0.005952478
3050	0.005488187
3100	0.005126287
3150	0.004809151
3200	0.004502112
3250	0.00415418
3300	0.003787971
3350	0.003416124
3400	0.003113261
3450	0.002884384
3500	0.002712438
3550	0.002539422
3600	0.00237917
3650	0.002194631
3700	0.002012283
3750	0.001832782
3800	0.001689359
3850	0.001575236
3900	0.001479248
3950	0.001389664
4000	0.001300905
4050	0.001202591
4100	0.001105537
4150	0.001018268
4200	0.0009528896
4250	0.0008975626
4300	0.0008479797
4350	0.0007998784
4400	0.0007507796
4450	0.0007006323
4500	0.0006531484
4550	0.0006124141
4600	0.0005790229
4650	0.0005500604
4700	0.0005230569
4750	0.0004954655
4800	0.0004675543
4850	0.0004399829
4900	0.0004158263
4950	0.0003958073
5000	0.000379317
5050	0.000363991
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5150	0.000326914
5200	0.0003052099
5250	0.0002787341
5300	0.0002529004
5350	0.0002277127
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5450	0.0001962121
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5550	0.0001737955
5600	0.0001624122
5650	0.0001488446
5700	0.0001373669
5750	0.0001263052
5800	0.000118011
5850	0.0001107766
5900	0.0001040599
5950	9.743528e-05
6000	9.092917e-05
6050	8.358866e-05
6100	7.668084e-05
6150	7.067427e-05
6200	6.663124e-05
6250	6.289564e-05
6300	5.925134e-05
6350	5.556742e-05
6400	5.188595e-05
6450	4.82808e-05
6500	4.499785e-05
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6900	2.485274e-05
6950	2.321983e-05
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7150	1.767707e-05
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7250	1.492001e-05
7300	1.383279e-05
7350	1.292017e-05
7400	1.210623e-05
7450	1.132202e-05
7500	1.054707e-05
7550	9.715013e-06
7600	8.917393e-06
7650	8.207151e-06
7700	7.665191e-06
7750	7.183995e-06
7800	6.733605e-06
7850	6.28961e-06
7900	5.843517e-06
7950	5.400064e-06
8000	4.987485e-06
8050	4.631344e-06
8100	4.329252e-06
8150	4.053774e-06
8200	3.789059e-06
8250	3.514384e-06
8300	3.240506e-06
8350	2.97398e-06
8400	2.74457e-06
8450	2.55368e-06
8500	2.392651e-06
8550	2.236074e-06
8600	2.087572e-06
8650	1.932625e-06
8700	1.782925e-06
8750	1.640221e-06
8800	1.518495e-06
8850	1.414591e-06
8900	1.322323e-06
8950	1.235313e-06
9000	1.150604e-06
9050	1.064332e-06
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9150	9.062673e-07
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9250	7.881537e-07
9300	7.37304e-07
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9550	5.100841e-07
9600	4.762568e-07
9650	4.458731e-07
9700	4.173919e-07

9750	3.89328e-07
9800	3.618379e-07
9850	3.353565e-07
9900	3.115734e-07
9950	2.908812e-07
10000	2.72852e-07
10050	2.560591e-07
10100	2.399571e-07
10150	2.235727e-07
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10250	1.916447e-07
10300	1.771989e-07
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10400	1.530117e-07
10450	1.431033e-07
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10550	1.245781e-07
10600	1.156529e-07
10650	1.069598e-07
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10750	9.230825e-08
10800	8.617471e-08
10850	8.050322e-08
10900	7.509164e-08
10950	6.984391e-08
11000	6.485112e-08
11050	6.010704e-08
11100	5.581675e-08
11150	5.197318e-08
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11250	4.54504e-08
11300	4.239434e-08
11350	3.944083e-08
11400	3.666186e-08
11450	3.409962e-08
11500	3.179003e-08
11550	2.970897e-08
11600	2.780238e-08
11650	2.596375e-08
11700	2.418523e-08
11750	2.242166e-08
11800	2.076144e-08
11850	1.922089e-08
11900	1.787872e-08
11950	1.669863e-08
12000	1.563109e-08
12050	1.457616e-08
12100	1.357108e-08
12150	1.257857e-08
12200	1.166228e-08
12250	1.081347e-08
12300	1.006783e-08
12350	9.396269e-09
12400	8.771853e-09
12450	8.174349e-09
12500	7.601774e-09
12550	7.047213e-09
12600	6.532476e-09
12650	6.066482e-09
12700	5.660365e-09
12750	5.286055e-09
12800	4.932805e-09
12850	4.593659e-09
12900	4.269012e-09
12950	3.961909e-09
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13150	2.985222e-09
13200	2.78297e-09
13250	2.585686e-09
13300	2.397497e-09
13350	2.220619e-09
13400	2.062279e-09
13450	1.921621e-09
13500	1.794885e-09
13550	1.674038e-09
13600	1.559484e-09
13650	1.447682e-09
13700	1.342319e-09
13750	1.244042e-09
13800	1.156301e-09
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13900	1.005284e-09
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14000	8.71927e-10
14050	8.090865e-10
14100	7.500927e-10
14150	6.960775e-10
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14250	6.044181e-10
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14400	4.882372e-10
14450	4.531424e-10
14500	4.20542e-10
14550	3.909988e-10
14600	3.644082e-10
14650	3.399606e-10
14700	3.170107e-10
14750	2.950093e-10
14800	2.74034e-10
14850	2.542529e-10
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14950	2.199535e-10
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15200	1.542263e-10
15250	1.430428e-10
15300	1.328613e-10
15350	1.236231e-10
15400	1.152384e-10
15450	1.074628e-10
15500	1.001369e-10
15550	9.307987e-11
15600	8.643043e-11
15650	8.022133e-11
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15750	6.946859e-11
15800	6.475765e-11
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15950	5.220101e-11
16000	4.848355e-11
16050	4.504715e-11
16100	4.191742e-11
16150	3.905278e-11
16200	3.641544e-11
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16350	2.932962e-11
16400	2.725622e-11
16450	2.535457e-11
16500	2.361988e-11
16550	2.202163e-11
16600	2.053411e-11
16650	1.91219e-11
16700	1.778375e-11
16750	1.651532e-11
16800	1.534103e-11
16850	1.426191e-11
16900	1.328186e-11
16950	1.238217e-11
17000	1.154456e-11
17050	1.074534e-11
17100	9.991647e-12
17150	9.280267e-12
17200	8.625375e-12
17250	8.022259e-12
17300	7.471921e-12
17350	6.963046e-12
17400	6.485993e-12
17450	6.034918e-12
17500	5.610294e-12
17550	5.212665e-12
17600	4.846061e-12
17650	4.509773e-12
17700	4.202515e-12
17750	3.916015e-12
17800	3.646416e-12
17850	3.391797e-12
17900	3.153009e-12
17950	2.931018e-12
18000	2.726724e-12
18050	2.539279e-12
18100	2.366499e-12
18150	2.20468e-12
18200	2.052159e-12
18250	1.907715e-12
18300	1.772623e-12
18350	1.647282e-12
18400	1.532661e-12
18450	1.427686e-12
18500	1.330678e-12
18550	1.239215e-12
18600	1.153143e-12
18650	1.071747e-12
18700	9.95956e-13
18750	9.257519e-13
18800	8.615377e-13
18850	8.024916e-13
18900	7.475822e-13
18950	6.959755e-13
19000	6.473806e-13
19050	6.016595e-13
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19150	5.199873e-13
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19400	3.635227e-13
19450	3.378828e-13
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19550	2.923009e-13
19600	2.722379e-13
19650	2.536004e-13
19700	2.36138e-13
19750	2.196676e-13
19800	2.042069e-13
19850	1.897861e-13
19900	1.765049e-13

19950	1.64317e-13	22500	4.173177e-15	25050	1.060597e-16	27600	2.689871e-18
20000	1.530857e-13	22550	3.880983e-15	25100	9.869274e-17	27650	2.503505e-18
20050	1.425919e-13	22600	3.610857e-15	25150	9.180308e-17	27700	2.330154e-18
20100	1.327482e-13	22650	3.361057e-15	25200	8.538599e-17	27750	2.168471e-18
20150	1.23457e-13	22700	3.129576e-15	25250	7.942476e-17	27800	2.017579e-18
20200	1.147592e-13	22750	2.913442e-15	25300	7.390683e-17	27850	1.876887e-18
20250	1.066595e-13	22800	2.711043e-15	25350	6.879327e-17	27900	1.746042e-18
20300	9.920999e-14	22850	2.52156e-15	25400	6.403887e-17	27950	1.624588e-18
20350	9.235737e-14	22900	2.345046e-15	25450	5.960169e-17	28000	1.511906e-18
20400	8.602136e-14	22950	2.181375e-15	25500	5.545549e-17	28050	1.407196e-18
20450	8.010628e-14	23000	2.030057e-15	25550	5.158261e-17	28100	1.309709e-18
20500	7.455491e-14	23050	1.889942e-15	25600	4.797892e-17	28150	1.218745e-18
20550	6.93252e-14	23100	1.759742e-15	25650	4.463521e-17	28200	1.133881e-18
20600	6.444187e-14	23150	1.637987e-15	25700	4.153994e-17	28250	1.054794e-18
20650	5.991241e-14	23200	1.523978e-15	25750	3.86665e-17	28300	9.812976e-19
20700	5.575181e-14	23250	1.417276e-15	25800	3.599105e-17	28350	9.130815e-19
20750	5.190919e-14	23300	1.318041e-15	25850	3.349281e-17	28400	8.497765e-19
20800	4.834216e-14	23350	1.22608e-15	25900	3.115917e-17	28450	7.909235e-19
20850	4.5004e-14	23400	1.141124e-15	25950	2.898295e-17	28500	7.360978e-19
20900	4.186901e-14	23450	1.062432e-15	26000	2.696037e-17	28550	6.849278e-19
20950	3.892894e-14	23500	9.892041e-16	26050	2.508534e-17	28600	6.372152e-19
21000	3.619201e-14	23550	9.206123e-16	26100	2.334781e-17	28650	5.927845e-19
21050	3.366212e-14	23600	8.564451e-16	26150	2.173297e-17	28700	5.515167e-19
21100	3.133258e-14	23650	7.964712e-16	26200	2.022785e-17	28750	5.132043e-19
21150	2.917698e-14	23700	7.407849e-16	26250	1.882155e-17	28800	4.776291e-19
21200	2.717079e-14	23750	6.891808e-16	26300	1.750872e-17	28850	4.445318e-19
21250	2.528816e-14	23800	6.414699e-16	26350	1.628586e-17	28900	4.136817e-19
21300	2.352107e-14	23850	5.972002e-16	26400	1.515094e-17	28950	3.849031e-19
21350	2.186806e-14	23900	5.559213e-16	26450	1.409924e-17	29000	3.580837e-19
21400	2.033554e-14	23950	5.172998e-16	26500	1.31238e-17	29050	3.331317e-19
21450	1.892197e-14	24000	4.811926e-16	26550	1.22159e-17	29100	3.099589e-19
21500	1.761842e-14	24050	4.475205e-16	26600	1.136905e-17	29150	2.884441e-19
21550	1.640795e-14	24100	4.162731e-16	26650	1.057771e-17	29200	2.684547e-19
21600	1.527821e-14	24150	3.873427e-16	26700	9.839588e-18	29250	2.498411e-19
21650	1.42166e-14	24200	3.605738e-16	26750	9.152543e-18	29300	2.324861e-19
21700	1.322147e-14	24250	3.356744e-16	26800	8.515323e-18	29350	2.163021e-19
21750	1.229145e-14	24300	3.124264e-16	26850	7.924584e-18	29400	2.012306e-19
21800	1.143076e-14	24350	2.906798e-16	26900	7.376195e-18	29450	1.872186e-19
21850	1.063634e-14	24400	2.70374e-16	26950	6.86557e-18	29500	1.742075e-19
21900	9.903186e-15	24450	2.514756e-16	27000	6.389115e-18	29550	1.621237e-19
21950	9.222261e-15	24500	2.339543e-16	27050	5.944028e-18	29600	1.508881e-19
22000	8.586109e-15	24550	2.177346e-16	27100	5.529244e-18	29650	1.404201e-19
22050	7.988027e-15	24600	2.026982e-16	27150	5.143566e-18	29700	1.306591e-19
22100	7.428512e-15	24650	1.886905e-16	27200	4.785987e-18	29750	1.215588e-19
22150	6.9069e-15	24700	1.756016e-16	27250	4.454181e-18	29800	1.130894e-19
22200	6.424967e-15	24750	1.633555e-16	27300	4.145822e-18	29850	1.052185e-19
22250	5.979379e-15	24800	1.519347e-16	27350	3.858471e-18	29900	9.791099e-20
22300	5.567165e-15	24850	1.413181e-16	27400	3.590279e-18	29950	9.112248e-20
22350	5.183386e-15	24900	1.314891e-16	27450	3.340021e-18	30000	8.480666e-20
22400	4.824239e-15	24950	1.2239e-16	27500	3.107021e-18		
22450	4.487569e-15	25000	1.139433e-16	27550	2.890597e-18		

E Exactly-N Points Probability Table

delta	probability
50	0.1157407
100	0.2954961
150	0.3321182
200	0.3866735
250	0.3208744
300	0.2636887
350	0.1462441
400	0.09665988
450	0.05979632
500	0.0933982
550	0.07949564
600	0.1015775
650	0.07985713
700	0.07107501
750	0.044094
800	0.03650522
850	0.03354724
900	0.03284479
950	0.03118512
1000	0.04832338
1050	0.04319938
1100	0.03379447
1150	0.01511773
1200	0.01478655
1250	0.0161609
1300	0.01731426
1350	0.01753176
1400	0.0165569
1450	0.01398602
1500	0.01067571
1550	0.01145659
1600	0.01579396
1650	0.0168085
1700	0.01852217
1750	0.01636457
1800	0.01403022
1850	0.009430557
1900	0.007133093
1950	0.005925601
2000	0.007191075
2050	0.006891007
2100	0.007634889
2150	0.00682614
2200	0.006065944
2250	0.004546057
2300	0.003755453
2350	0.003443962
2400	0.003404185
2450	0.003305004
2500	0.003718185
2550	0.003342113
2600	0.002768666
2650	0.001932947
2700	0.001793756
2750	0.001802519
2800	0.001832617
2850	0.001815522
2900	0.00171287
2950	0.001488236
3000	0.001208332
3050	0.001085162
3100	0.001146199
3150	0.001155619
3200	0.00119756
3250	0.001109017
3300	0.0009907109
3350	0.0007791216
3400	0.0006414405
3450	0.0005628691
3500	0.0005871278
3550	0.0005708988
3600	0.0005854806
3650	0.0005431186
3700	0.0004867248
3750	0.0003982062
3800	0.0003390314
3850	0.0003107285
3900	0.0003016732
3950	0.0002923532
4000	0.0002961168
4050	0.0002700155
4100	0.0002334307
4150	0.0001888718
4200	0.0001716768
4250	0.0001648918
4300	0.0001617214
4350	0.0001571136
4400	0.0001478753
4450	0.0001315945
4500	0.0001120137
4550	9.87467e-05
4600	9.472925e-05
4650	9.18331e-05
4700	9.090145e-05
4750	8.556333e-05
4800	7.813801e-05
4850	6.663317e-05
4900	5.739046e-05
4950	5.118927e-05
5000	4.965707e-05
5050	6.254366e-05
5100	8.476379e-05
5150	8.645039e-05
5200	8.933397e-05
5250	7.536191e-05
5300	6.374426e-05
5350	4.642132e-05
5400	4.046756e-05
5450	3.689655e-05
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